

CYCLE-DECORATED RIBBON COMPLEXES: CUT COPRODUCTS AND ALTERNATING-FENCE POSITIVITY

PYUYI CHUFENG HUANG

ABSTRACT. Let $\alpha \models n$. We define a two-variable specialization $Z_\alpha(t, q)$ of the ribbon basis of noncommutative symmetric functions from cycle enumerators of an ordinary permutation and a rooted permutation, with q recording reflection length. We realize $n!Z_\alpha(t, q)$ as the shifted bigraded Euler characteristic of an $\mathfrak{S}_n$ -equivariant ordered-set-partition complex. When α has at most one odd part, each total decoration determines a set of simultaneous factorization cuts, and its fiber is the classical ribbon complex indexed by that cut set. This gives explicit nonnegative ribbon expansions for every bigraded homology representation.

Organizing the complexes on labelled finite sets yields a counital differential graded comonoid in the Cauchy monoidal category of species, whose cut coproduct is compatible with the fiber decomposition. At a factorization cut, the induced map on top homology is injective; its cokernel has the near-concatenation ribbon character, and the kernel of the aggregate reduced cut coproduct on homology is H_1 .

For the zigzag compositions δ_n , $Z_{\delta_n}(t, -1)$ is the order polynomial of the alternating fence. We prove

$$n!Z_{\delta_n}(t, -u) \in \mathbb{N}[t, u]$$

using coefficientwise nonnegative recurrences derived from a Riccati equation. The same cut data define a nonnegative factorization defect $d = \lceil n/2 \rceil - k - j$, which controls the homological support and makes the Euler sign constant on each defect layer. We also determine the full defect-zero edge by explicit Frobenius-character formulas.

1. INTRODUCTION

Ribbon functions connect Boolean inclusion–exclusion, descent representations, and rank-selected homology. For a nonempty composition $\alpha \models n$, the ribbon element in the algebra of noncommutative symmetric functions is

$$R_\alpha = \sum_{\beta \succeq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} H_\beta.$$

Under abelianization, R_α maps to the ribbon Schur function r_α , which is the Frobenius characteristic of the corresponding descent representation. We refine this picture by two cycle statistics and then study the zigzag compositions that encode alternating fences.

For $m \geq 0$, write

$$m^+ = \left\lceil \frac{m}{2} \right\rceil, \quad m^- = \left\lfloor \frac{m}{2} \right\rfloor,$$

and set

$$h_m(t, q) = \frac{1}{m!} \prod_{i=1}^{m^-} (t + i) \prod_{j=0}^{m^+ - 1} (t + jq).$$

The second product is the cycle enumerator of a permutation in $\mathfrak{S}_{m^+}$, with q recording its reflection length $m^+ - c(\sigma)$. The first product is obtained from the cycle enumerator of a permutation on $m^- + 1$ letters with one distinguished letter. Since the complete functions freely generate NSym, the assignments $H_m \mapsto h_m(t, q)$ define an algebra homomorphism $\phi_{t,q}$. We write

$$Z_\alpha(t, q) = \phi_{t,q}(R_\alpha).$$

2020 *Mathematics Subject Classification*. Primary 05E10; Secondary 05E05, 05A15, 06A07.

Key words and phrases. noncommutative symmetric functions, ribbon Schur functions, equivariant homology, combinatorial species, cut coproduct, reflection length, alternating fences, order polynomials.

Main results. Our first result gives an equivariant homological realization of the ribbon evaluation. The chain groups are spanned by ordered set partitions whose blocks carry an ordinary permutation and a rooted permutation. Adjacent blocks are merged by direct sum in the ordinary component and by gluing the two root cycles in the rooted component. Suppose that α has at most one odd part. For a total decoration θ , let $F_\alpha(\theta)$ be the set of cuts at which both components factor. The fiber with total decoration θ is the classical ribbon complex with cut set $F_\alpha(\theta)$. Consequently, [Theorem 3.11](#) gives

$$(1) \quad \text{ch}_{t,q} H_k(C_{\bullet}^{t,q}(\alpha)) = \sum_{\substack{\theta \in \mathcal{D}_n \\ |F_\alpha(\theta)|=k-1}} t^{\deg_t \theta} q^{\deg_q \theta} r_{\gamma_\alpha(\theta)}.$$

Thus every bigraded homology representation is determined explicitly and is Schur-positive. Grouping by exact cut set gives the nonnegative multiplicity polynomials in [Corollary 3.12](#).

The same factorization data define a cut coproduct. After the complexes are organized as a species on finite label sets, the coproduct makes their direct sum a counital differential graded comonoid in the Cauchy monoidal category. It respects the total-decoration fibers. At every factorization cut, the aggregate label-set component induces a canonical injection of the corresponding ribbon homology module into the induction product of the two factor modules; the cokernel has the near-concatenation ribbon character. The kernel on homology of the aggregate reduced coproduct is exactly H_1 ; see [Theorem 3.14](#) and [Corollaries 3.15](#) and [3.16](#).

Our second result concerns the zigzag compositions $\delta_{2N} = (2, \dots, 2)$ and $\delta_{2N+1} = (2, \dots, 2, 1)$. At $q = -1$, their evaluations are the order polynomials of alternating fences. Retaining reflection length by writing $q = -u$, [Theorem 4.3](#) proves

$$(2) \quad n! Z_{\delta_n}(t, -u) \in \mathbb{N}[t, u].$$

The proof starts from the hypergeometric differential equation for the even kernel and converts it into two simultaneous recurrences with nonnegative coefficients. The odd case follows from a first-order relation between the even and odd kernels.

The same total decoration controls the two parts of the paper. Its factorization cuts select the ribbon complex occurring in homology, while the excess of ordinary cycles over the factorization intervals determines the sign after the substitution $q = -u$. More precisely, if $\theta = (\sigma, \tau)$ contributes in homological degree k and q -degree j , then

$$\text{fd}_n(\theta) = c(\sigma) - |F_{\delta_n}(\theta)| - 1 = \left\lceil \frac{n}{2} \right\rceil - k - j \geq 0.$$

The differential preserves total decoration and therefore splits the zigzag complex by this factorization defect. On each defect layer the shifted Euler sign is constant, and [Theorem 4.5](#) gives the support condition

$$H_k^{i,j}(n) \neq 0 \implies k + j \leq \left\lceil \frac{n}{2} \right\rceil \leq i + j.$$

The defect-zero edge of this region is computed in [Corollary 4.7](#) by explicit representation-valued formulas. The virtual Euler characteristic is not Schur-positive in general, already for $n = 4$, so the scalar positivity in (2) does not arise from coefficientwise positivity in the representation ring.

The Riccati recurrence also gives positive integral logarithmic coefficients for the even zigzag series. At $u = 1$, they specialize to scaled order polynomials of alternating circular fences; see [Proposition 4.8](#).

Relation to earlier work. Up to augmentation and homological reindexing, the undecorated complex is the ordered-set-partition complex Δ_α of Ehrenborg and Jung. They identify it with a rank-selected Boolean complex, prove concentration of its homology, construct a basis indexed by exact descent composition, and identify the top homology with the border-strip Specht module [[5](#), [Theorem 4.2](#), [Theorem 6.6](#), and [Proposition 7.3](#)]. Bergeron and Krob constructed acyclic complexes related to ribbon elements in noncommutative symmetric functions [[4](#)]. After choosing an n -dimensional vector space with an ordered basis, the multidegree $(1, \dots, 1)$ part of VandeBogert's refinement complex for its symmetric algebra has the same ordered-set-partition basis and adjacent multiplication maps, up to homological reindexing [[18](#), [Definition A.7](#) and [Example A.8](#)]. In the present complex, these classical ribbon complexes occur as fibers indexed by total permutation decorations, and the simultaneous factorization cuts determine both the fiber and its bidegree.

Chain-level categorifications of the ribbon concatenation and near-concatenation identity occur in VandeBogert's ribbon Schur functors and, for projective 0-Hecke modules, in the canonical split exact sequences of Almousa and Lu [3, 18]. Here the injection in Corollary 3.15 is induced by a coassociative label-set cut coproduct on $\mathfrak{S}_n$ -equivariant complexes. It restricts to each total-decoration fiber and is compatible with simultaneous factorization at every cut. Hersh and Sundaram construct ribbon bases for rank-selected homology and Whitney homology of geometric lattices [9, Theorems 5.29–5.30].

Gessel and Reutenauer jointly enumerate cycle structure and descent sets of a single permutation [8]. Novelli, Thibon, and Toumazet lift the cycle-index polynomials of symmetric groups to bases of quasisymmetric and noncommutative symmetric functions and obtain a product formula and a combinatorial-complex recurrence [14]. In our construction, cycle statistics are carried by the block decorations, while the descent set is supplied by the homology of the ribbon fiber. Direct-sum factorization of ordinary permutations is standard; see [10, Equation (34), Proposition 3.3, and the discussion before Theorem 4.7]. The construction below additionally uses a rooted component and requires simultaneous factorization of the two components at the same cuts.

Kreweras's determinant for skew plane partitions [13], in the form used by Ferroni, Morales, and Panova, gives the alternating-fence specialization below. Ferroni, Morales, and Panova prove coefficientwise positivity for fence and circular-fence order polynomials [6], and Kahane gives a permutation statistic for the coefficients of every fence order polynomial [12, Theorem 3.8]. The author's earlier preprint gives a different record model for the same one-variable specialization and for circular fences [11]. These results concern $u = 1$; the refinement here retains reflection length and relates its sign to the homological factorization defect.

Organization and conventions. Section 2 defines the two-variable ribbon evaluation. Section 3 constructs the decorated complex and computes its equivariant homology. Section 4 proves the alternating-fence specialization, bivariate positivity, the defect decomposition, the support and defect-zero formulas, and the logarithmic consequence.

All vector spaces are over a field $\mathbb{K}$ of characteristic zero, and $\mathbb{N}$ includes zero. For $m \geq 0$, write $[m] = \{1, \dots, m\}$, with $[0] := \emptyset$; an integer interval $\{a, \dots, b\}$ is empty when $a > b$. Polynomial characters take values in $\mathbb{Q}[t, q]$. Chain complexes are homologically graded. The superscript t, q records the two weight gradings; the differential does not depend on numerical values of t and q . Compositions in Section 3 are nonempty. Empty internal alphabets and cut sets are allowed. Throughout, $\mathfrak{S}_n$ acts on the labels of ordered set partitions and fixes the decoration bases.

2. RIBBON EVALUATIONS WEIGHTED BY CYCLE STATISTICS

The ribbon basis turns coarsenings into inclusion–exclusion. We evaluate the complete generators by two cycle enumerators.

2.1. Compositions, ribbons, and NSym. A *composition* of a positive integer n , denoted $\alpha \models n$, is a sequence $\alpha = (\alpha_1, \dots, \alpha_k)$ of positive integers with

$$|\alpha| := \alpha_1 + \dots + \alpha_k = n.$$

Its length is $\ell(\alpha) := k$. We also use the empty composition $\emptyset \models 0$, of length zero.

A composition γ is a *coarsening* of α if it can be obtained from α by repeatedly replacing adjacent parts by their sum. Write $\text{Coars}(\alpha)$ for all coarsenings of α , including α .

Let $\mathcal{R} := \mathbb{Q}[t, q]$. The algebra of noncommutative symmetric functions over $\mathcal{R}$ is the free associative algebra

$$\text{NSym}_{\mathcal{R}} = \mathcal{R}\langle H_1, H_2, \dots \rangle,$$

where $H_0 := 1$ and $\deg H_m = m$. For $\alpha = (\alpha_1, \dots, \alpha_k)$, set $H_\alpha = H_{\alpha_1} \cdots H_{\alpha_k}$, with $H_\emptyset = 1$. We use the ribbon normalization

$$(3) \quad R_\alpha = \sum_{\gamma \in \text{Coars}(\alpha)} (-1)^{\ell(\alpha) - \ell(\gamma)} H_\gamma, \quad R_\emptyset = 1.$$

These are standard conventions; see [7]. Under the canonical abelianization $\text{NSym} \rightarrow \text{Sym}$, R_α maps to the ribbon Schur function r_α .

2.2. The two-variable character. For $m \geq 0$, put

$$m^+ = \left\lceil \frac{m}{2} \right\rceil, \quad m^- = \left\lfloor \frac{m}{2} \right\rfloor.$$

Thus $m^+ + m^- = m$. Define

$$(4) \quad h_m(t, q) = \frac{1}{m!} \left(\prod_{i=1}^{m^-} (t+i) \right) \left(\prod_{j=0}^{m^+-1} (t+jq) \right), \quad h_0(t, q) = 1.$$

Definition 2.1 (Two-variable character). Let $\phi_{t,q} : \text{NSym}_{\mathcal{R}} \rightarrow \mathcal{R}$ be the unique unital $\mathcal{R}$ -algebra homomorphism determined by

$$\phi_{t,q}(H_m) = h_m(t, q) \quad (m \geq 1).$$

For a composition α , define

$$(5) \quad Z_\alpha(t, q) = \phi_{t,q}(R_\alpha), \quad Z_\emptyset(t, q) = 1.$$

The map exists uniquely because $\text{NSym}_{\mathcal{R}}$ is freely generated by the H_m . Here a character is a unital algebra homomorphism, as usual in combinatorial Hopf algebra theory [2].

The unsigned Stirling cycle enumerator [17, Section 1.3], after homogenization, gives for $r \geq 1$

$$(6) \quad \sum_{\sigma \in \mathfrak{S}_r} q^{r-c(\sigma)} t^{c(\sigma)} = \prod_{j=0}^{r-1} (t+jq),$$

where $c(\sigma)$ counts all cycles, including one-cycles. Consequently,

$$m! \phi_{t,q}(H_m) = \left(\prod_{i=1}^{m^-} (t+i) \right) \sum_{\sigma \in \mathfrak{S}_{m^+}} q^{m^+-c(\sigma)} t^{c(\sigma)}.$$

Thus both factors in the definition have direct cycle interpretations, and $r - c(\sigma)$ is the reflection length of σ with respect to all transpositions.

At $q = -1$, the two products in (4) combine to give the falling factorial $(t + m^-)^{\overline{m}}$, where $x^{\overline{m}} = x(x-1) \cdots (x-m+1)$. Hence

$$\phi_{t,-1}(H_m) = \binom{t+m^-}{m} = \binom{t+\lfloor m/2 \rfloor}{m}.$$

Thus, writing $\phi_t = \phi_{t,-1}$,

$$(7) \quad \phi_t(H_{2a}) = \binom{t+a}{2a},$$

$$(8) \quad \phi_t(H_{2a+1}) = \binom{t+a}{2a+1}.$$

Applying $\phi_{t,q}$ to (3) gives

$$(9) \quad Z_\alpha(t, q) = \sum_{\gamma \in \text{Coars}(\alpha)} (-1)^{\ell(\alpha) - \ell(\gamma)} \prod_{j=1}^{\ell(\gamma)} h_{\gamma_j}(t, q).$$

Reversal bijects the coarsenings of α with those of α^{rev} , preserves their lengths, and reverses their parts. Since the scalar factors in (9) commute,

$$(10) \quad Z_{\alpha^{\text{rev}}}(t, q) = Z_\alpha(t, q).$$

For later use, define the zigzag compositions

$$\delta_0 = \emptyset, \quad \delta_{2N} = \underbrace{(2, \dots, 2)}_{N \text{ parts}}, \quad \delta_{2N+1} = \underbrace{(2, \dots, 2)}_{N \text{ parts}}, 1).$$

By (10), the last composition may equivalently be written $(1, 2, \dots, 2)$.

3. THE DECORATED RIBBON COMPLEX

Ordered set partitions realize the alternating sum over coarsenings at chain level. After recalling the undecorated ribbon complex, we establish the factorization criterion and apply it to rooted permutation decorations.

Throughout this section, $\alpha \models n$ is nonempty, so $n \geq 1$. All representations are finite-dimensional.

3.1. The classical ribbon complex.

Definition 3.1 (Cut set and coarsening). For a nonempty composition $\alpha = (\alpha_1, \dots, \alpha_r) \models n$, define its cut set by

$$D(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \dots + \alpha_{r-1}\} \subseteq [n-1].$$

For the coarsening relation defined in Section 2, write $\beta \succeq \alpha$. Equivalently,

$$\beta \succeq \alpha \iff D(\beta) \subseteq D(\alpha).$$

Write

$$\text{Coars}_k(\alpha) := \{\beta \succeq \alpha : \ell(\beta) = k\}.$$

Thus the interval of coarsenings of α is canonically a Boolean lattice on the $r-1$ cuts in $D(\alpha)$. If $\beta = (\beta_1, \dots, \beta_k) \succeq \alpha$ and $1 \leq i < k$, let

$$m_i \beta := (\beta_1, \dots, \beta_{i-1}, \beta_i + \beta_{i+1}, \beta_{i+2}, \dots, \beta_k).$$

Deleting one more cut shows that $m_i \beta \succeq \alpha$.

For $S \subseteq D(\alpha)$, let γ_S denote the unique composition of n with $D(\gamma_S) = S$. Ordered set partitions.

Definition 3.2 (Ordered set partitions). Let I be a finite set with $|I| = n$, and let $\beta = (\beta_1, \dots, \beta_k) \models n$. Define

$$\text{OSP}_\beta[I] := \left\{ (B_1, \dots, B_k) : \begin{array}{l} B_i \neq \emptyset, B_i \cap B_j = \emptyset \ (i \neq j), \\ B_1 \sqcup \dots \sqcup B_k = I, |B_i| = \beta_i \end{array} \right\}.$$

For $I = [n]$, let

$$M^\beta := \mathbb{K}[\text{OSP}_\beta[n]]$$

be the permutation module obtained by linearly extending the natural action of $\mathfrak{S}_n$ on labels.

The stabilizer of the standard ordered partition is the corresponding Young subgroup. Hence, for every $\beta \models n$,

$$(11) \quad |\text{OSP}_\beta[n]| = \binom{n}{\beta_1, \dots, \beta_k} = \frac{n!}{\beta_1! \dots \beta_k!},$$

$$M^\beta \cong \text{Ind}_{\mathfrak{S}_{\beta_1} \times \dots \times \mathfrak{S}_{\beta_k}}^{\mathfrak{S}_n} \mathbf{1}, \quad \text{ch}(M^\beta) = h_\beta := h_{\beta_1} \dots h_{\beta_k}.$$

The chain complex. For $\mathbf{B} = (B_1, \dots, B_k) \in \text{OSP}_\beta[n]$, define

$$d_i \mathbf{B} := (B_1, \dots, B_{i-1}, B_i \sqcup B_{i+1}, B_{i+2}, \dots, B_k) \in \text{OSP}_{m_i \beta}[n].$$

Linear extension gives an $\mathfrak{S}_n$ -equivariant map

$$d_i^\beta : M^\beta \longrightarrow M^{m_i \beta}.$$

Definition 3.3 (Ribbon complex). For a nonempty composition $\alpha \models n$ with $r = \ell(\alpha)$, set

$$C_k(\alpha) := \bigoplus_{\beta \in \text{Coars}_k(\alpha)} M^\beta, \quad 1 \leq k \leq r,$$

and set $C_k(\alpha) = 0$ outside this range. On the summand M^β , define

$$(12) \quad \partial_k|_{M^\beta} := \sum_{i=1}^{k-1} (-1)^{i-1} d_i^\beta.$$

We call $(C_\bullet(\alpha), \partial)$ the *ribbon complex* of α , and we place $C_k(\alpha)$ in homological degree k . For Euler characteristics we use the globally shifted sign $(-1)^{r-k}$, which agrees with the usual ribbon expansion.

Associativity of disjoint union gives the usual face identities

$$d_i d_j = d_{j-1} d_i \quad (1 \leq i < j \leq k-1).$$

The two orders of deleting any pair of cuts therefore give equal maps with opposite signs in (12). Hence $\partial^2 = 0$.

Homology. Let $\mathcal{B}_n$ be the Boolean lattice of subsets of $[n]$. For $D \subseteq [n-1]$, write

$$\mathcal{B}_n(D) := \{S \subseteq [n] : |S| \in D\}$$

for its proper rank-selected subposet.

Proposition 3.4 (Known ribbon-complex homology). *Let $\alpha \models n$ be nonempty. The map*

$$(B_1, \dots, B_k) \mapsto B_1 \subset B_1 \sqcup B_2 \subset \dots \subset B_1 \sqcup \dots \sqcup B_{k-1}$$

identifies $C_\bullet(\alpha)$, after the degree shift $C_k \leftrightarrow \tilde{C}_{k-2}$, with the augmented simplicial chain complex of $\Delta(\mathcal{B}_n(D(\alpha)))$. The identification is $\mathfrak{S}_n$ -equivariant. The homology is concentrated in degree $\ell(\alpha)$, and

$$(13) \quad \begin{aligned} \text{ch } H_{\ell(\alpha)}(C_\bullet(\alpha)) &= r_\alpha, \\ \dim H_{\ell(\alpha)}(C_\bullet(\alpha)) &= \#\{\pi \in \mathfrak{S}_n : \text{Des}(\pi) = D(\alpha)\}. \end{aligned}$$

Proof. The inverse sends a chain $S_1 \subset \dots \subset S_{k-1}$ to the successive differences $S_1, S_2 \setminus S_1, \dots, [n] \setminus S_{k-1}$. Merging adjacent blocks deletes the corresponding rank, so this is an equivariant chain identification; when $D(\alpha) = \emptyset$, we use $\tilde{C}_{-1}(\emptyset) = \mathbb{K}$. The homology concentration, descent basis, and ribbon character follow from [5, Theorem 4.2, Theorem 6.6, and Proposition 7.3]; see also [16, Theorem 4.3] and [19, Theorem 3.4.4]. $\square$

3.2. Factorization along cuts. We now give a criterion that decomposes a decorated ribbon complex into subcomplexes with fixed total decoration. Fix a nonempty composition $\alpha \models n$. For each integer m that occurs as a block size in a coarsening of α , choose a finite set $\mathcal{A}_m$. Equip these sets with a bidegree

$$\text{bideg}: \mathcal{A}_m \longrightarrow \mathbb{N}^2,$$

whose coordinates we denote by $\deg_t$ and $\deg_q$. Suppose that whenever a, b are adjacent block sizes in such a coarsening there is a product

$$\diamond: \mathcal{A}_a \times \mathcal{A}_b \longrightarrow \mathcal{A}_{a+b}.$$

We require

$$\text{bideg}(x \diamond y) = \text{bideg}(x) + \text{bideg}(y)$$

and $(x \diamond y) \diamond z = x \diamond (y \diamond z)$ whenever the three consecutive block sizes occur in a coarsening of α . Thus for every $\beta = (\beta_1, \dots, \beta_k) \succeq \alpha$ there is an unambiguous total product

$$\mu_\beta: \mathcal{A}_{\beta_1} \times \dots \times \mathcal{A}_{\beta_k} \longrightarrow \mathcal{A}_n,$$

where $\mu_{(n)} = \text{id}_{\mathcal{A}_n}$.

Decorate an ordered set partition of type β by one element of $\mathcal{A}_{\beta_i}$ on each block. Merging adjacent blocks and multiplying their decorations defines a complex $C_\bullet^{\mathcal{A}}(\alpha)$ with

$$C_k^{\mathcal{A}}(\alpha) = \bigoplus_{\substack{\beta \succeq \alpha \\ \ell(\beta) = k}} \mathbb{K}[\text{OSP}_\beta[n]] \otimes \mathbb{K}[\mathcal{A}_{\beta_1}] \otimes \dots \otimes \mathbb{K}[\mathcal{A}_{\beta_k}],$$

with the signs of (12). The group $\mathfrak{S}_n$ acts on the labels of the ordered set partition and fixes all decorations. Associativity gives the face identities and hence makes the boundary square to zero.

Definition 3.5 (Factorization condition). The preceding decoration system has *unique factorization along cuts with respect to α* if every map μ_β is injective and, for each $\theta \in \mathcal{A}_n$, there is a set $F_\alpha(\theta) \subseteq D(\alpha)$ such that

$$(14) \quad \theta \in \text{im } \mu_\beta \iff D(\beta) \subseteq F_\alpha(\theta) \quad (\beta \succeq \alpha).$$

Write $\gamma_\alpha(\theta)$ for the composition of n with cut set $F_\alpha(\theta)$.

Condition (14) states that the factors at all selected cuts exist simultaneously and are unique. For a finite-dimensional bigraded $\mathfrak{S}_n$ -module V , write

$$\mathrm{ch}_{t,q} V := \sum_{d,e} \mathrm{ch}(V_{d,e}) t^d q^e.$$

For a finite-dimensional bigraded vector space V , write

$$\mathrm{Hilb}_{t,q} V := \sum_{d,e} \dim(V_{d,e}) t^d q^e.$$

Lemma 3.6 (Fiber decomposition by total decorations). *If $\mathcal{A}$ has unique factorization along cuts with respect to α , then the total product gives a canonical bidegree-preserving, $\mathfrak{S}_n$ -equivariant chain isomorphism*

$$(15) \quad C_{\bullet}^{\mathcal{A}}(\alpha) \cong \bigoplus_{\theta \in \mathcal{A}_n} C_{\bullet}(\gamma_{\alpha}(\theta)).$$

The summand indexed by θ is placed in bidegree $\mathrm{bideg}(\theta)$, and

$$(16) \quad \mathrm{ch}_{t,q} H_k(C_{\bullet}^{\mathcal{A}}(\alpha)) = \sum_{\substack{\theta \in \mathcal{A}_n \\ |F_{\alpha}(\theta)|=k-1}} t^{\deg_t \theta} q^{\deg_q \theta} r_{\gamma_{\alpha}(\theta)}.$$

In particular, every bigraded homology Frobenius characteristic admits an expansion with coefficients in $\mathbb{N}[t, q]$ in ribbon Schur functions.

Proof. Associativity implies that every face map preserves the total product. On a basis element of type β , define

$$\Psi(\mathbf{B}; x_1, \dots, x_k) = \mathbf{B}$$

in the summand indexed by $\theta = \mu_{\beta}(x_1, \dots, x_k)$. Fixing θ , injectivity of μ_{β} gives at most one tuple $(x_1, \dots, x_k)$, and (14) says that this tuple exists exactly when $D(\beta) \subseteq F_{\alpha}(\theta)$. These β are precisely the coarsenings of $\gamma_{\alpha}(\theta)$, so restriction of Ψ to the subcomplex with total decoration θ is a basis bijection onto $C_{\bullet}(\gamma_{\alpha}(\theta))$.

Merging adjacent blocks replaces (x_i, x_{i+1}) by $x_i \diamond x_{i+1}$ and does not change θ ; under Ψ it is therefore exactly the ordinary adjacent-union face. Thus Ψ is a chain isomorphism. Degree additivity and the fact that $\mathfrak{S}_n$ acts only on the block labels prove the grading and equivariance assertions. Summing the fibers proves (15), and (16) follows from Proposition 3.4. $\square$

3.3. Permutation decorations. For $\sigma \in \mathfrak{S}_a$ and $\tau \in \mathfrak{S}_b$, define their direct sum $\sigma \oplus \tau \in \mathfrak{S}_{a+b}$ by

$$(\sigma \oplus \tau)(j) = \begin{cases} \sigma(j), & 1 \leq j \leq a, \\ a + \tau(j - a), & a < j \leq a + b. \end{cases}$$

Direct inspection gives the equalities

$$(17) \quad (\sigma \oplus \tau) \oplus v = \sigma \oplus (\tau \oplus v), \quad c(\sigma \oplus \tau) = c(\sigma) + c(\tau).$$

The decoration product. The two components of $\mathcal{D}_m$ account for the two factors in (4): ordinary permutations record the q -weighted cycle statistic, while rooted permutations record the second cycle count.

For $s \geq 0$, let

$$\mathfrak{R}_s := \mathfrak{S}_{\{*, 1, \dots, s\}}$$

be the set of permutations of $s + 1$ letters with distinguished letter $*$. We call its elements *rooted permutations*. The cycle containing $*$ has a unique cyclic notation $(*, a_1, \dots, a_u)$ beginning at the root. In particular, $\mathfrak{R}_0$ consists of the single rooted permutation $(*)$.

Definition 3.7 (Product of rooted permutations). Let $\tau \in \mathfrak{R}_a$ and $v \in \mathfrak{R}_b$. Write their root cycles on the original alphabets as

$$(*, a_1, \dots, a_u), \quad (*, b_1, \dots, b_v),$$

and then shift every nonroot letter of v by a . Define $\tau \star v \in \mathfrak{R}_{a+b}$ by replacing the two root cycles with

$$(*, a_1, \dots, a_u, a + b_1, \dots, a + b_v)$$

and retaining every nonroot cycle of τ , together with every nonroot cycle of v after the same shift by a .

The sequences following the roots concatenate, and the nonroot cycles form a disjoint union. Therefore $\star$ is associative, the identity permutation on $\{*\}$ is its unit, and

$$c(\tau \star v) = c(\tau) + c(v) - 1.$$

The last identity holds because the two root cycles become one and every other cycle remains unchanged.

For ordinary permutations we use the direct sum $\sigma \oplus \sigma'$ defined above. Retaining m^+ and m^- from Section 2, put

$$\mathcal{D}_m := \mathfrak{S}_{m^+} \times \mathfrak{R}_{m^-} \quad (m \geq 1).$$

Give a decoration $\xi = (\sigma, \tau) \in \mathcal{D}_m$ the bidegree

$$(18) \quad \deg_q(\xi) := m^+ - c(\sigma), \quad \deg_t(\xi) := c(\sigma) + c(\tau) - 1.$$

Thus $\deg_q(\xi)$ is the reflection length of the ordinary component, while $\deg_t(\xi)$ counts the ordinary cycles together with the nonroot cycles of the rooted component.

If a and b are not both odd, then $(a+b)^+ = a^+ + b^+$ and $(a+b)^- = a^- + b^-$. In this case define

$$(19) \quad (\sigma, \tau) \diamond (\sigma', \tau') := (\sigma \oplus \sigma', \tau \star \tau') \in \mathcal{D}_{a+b}.$$

Let $A_m^{\text{dec}} = \mathbb{K}[\mathcal{D}_m]$ for $m \geq 1$, let $A_0^{\text{dec}} = \mathbb{K}1$, with 1 a two-sided unit of bidegree $(0, 0)$, and extend (19) bilinearly by

$$(20) \quad A_a^{\text{dec}} \diamond A_b^{\text{dec}} = 0 \quad \text{when } a \text{ and } b \text{ are both odd.}$$

The product $\diamond$ makes $A^{\text{dec}} = \bigoplus_{m \geq 0} A_m^{\text{dec}}$ a connected bigraded associative algebra. This algebra is generally noncommutative. On every nonzero product, both degrees in (18) are additive:

$$\deg_q(\xi \diamond \xi') = \deg_q(\xi) + \deg_q(\xi'), \quad \deg_t(\xi \diamond \xi') = \deg_t(\xi) + \deg_t(\xi').$$

Indeed, on three factors containing at most one odd size, associativity follows from (17) and the associativity of $\star$. If at least two sizes are odd, both parenthesizations vanish by (20). The cycle formulas and the equalities $(a+b)^+ = a^+ + b^+$ and $(a+b)^- = a^- + b^-$ for every nonzero product prove degree additivity.

Let

$$A^{\text{even}} = \bigoplus_{m \text{ even}} A_m^{\text{dec}}, \quad A^{\text{odd}} = \bigoplus_{m \text{ odd}} A_m^{\text{dec}}.$$

Then $A^{\text{dec}} = A^{\text{even}} \oplus A^{\text{odd}}$, and A^{odd} is a square-zero A^{even} -bimodule.

The decorated complex.

Definition 3.8 (Ordered set partitions with permutation decorations). For an arbitrary composition $\beta = (\beta_1, \dots, \beta_k) \models n$, let

$$\text{ROSP}_\beta[n] := \left\{ (B_1, \dots, B_k; \xi_1, \dots, \xi_k) : \begin{array}{l} (B_1, \dots, B_k) \in \text{OSP}_\beta[n], \\ \xi_i \in \mathcal{D}_{\beta_i} \end{array} \right\}.$$

Let $M_{t,q}^\beta$ be its linear span, bigraded by the sums of the local t - and q -degrees, with $\mathfrak{S}_n$ acting on block labels and fixing the abstract decorations. Define linearly

$$d_i^{t,q}(\mathbf{B}; \xi_1, \dots, \xi_k) := \begin{cases} 0, & \beta_i \text{ and } \beta_{i+1} \text{ are both odd,} \\ (d_i \mathbf{B}; \xi_1, \dots, \xi_i \diamond \xi_{i+1}, \dots, \xi_k), & \text{otherwise.} \end{cases}$$

Thus every nonzero face sends a basis element to a basis element.

For every nonempty composition $\alpha \models n$, set

$$C_k^{t,q}(\alpha) := \bigoplus_{\substack{\beta \succeq \alpha \\ \ell(\beta) = k}} M_{t,q}^\beta, \quad \partial_k^{t,q} := \sum_{i=1}^{k-1} (-1)^{i-1} d_i^{t,q}.$$

Then $(\partial^{t,q})^2 = 0$, and the differential has bidegree $(0, 0)$. Indeed, associativity of union and $\diamond$ gives the face identities, so the standard alternating cancellation gives $(\partial^{t,q})^2 = 0$. The cycle formulas show that the differential has bidegree $(0, 0)$. Thus $C_\bullet^{t,q}(\alpha)$ is a bigraded complex of $\mathfrak{S}_n$ -modules.

The local bigraded enumerator is

$$Q_m(t, q) := \sum_{(\sigma, \tau) \in \mathcal{D}_m} q^{m^+ - c(\sigma)} t^{c(\sigma) + c(\tau) - 1}.$$

For every $m \geq 1$,

$$(21) \quad Q_m(t, q) = \prod_{i=0}^{m^+ - 1} (t + iq) \prod_{j=1}^{m^-} (t + j).$$

In particular,

$$(22) \quad Q_m(t, -1) = t^{m^+} \prod_{j=1}^{m^-} (t + j) = (t + m^-)^m = m! \binom{t + m^-}{m}.$$

Indeed, the ordinary factor is the classical cycle enumerator (6). For a rooted permutation on $s + 1$ letters, the cycle containing the root always contributes a factor t . Removing this factor from the usual cycle enumerator leaves $\prod_{j=1}^s (t + j)$. This proves (21). At $q = -1$, the factors combine to give (22).

Choosing the ordered set partition by (11) and then its block decorations shows that the bigraded Hilbert polynomial of the k -th chain group is

$$(23) \quad \Phi_{\alpha, k}(t, q) = \sum_{\substack{\beta \succeq \alpha \\ \ell(\beta) = k}} \binom{n}{\beta_1, \dots, \beta_k} \prod_{j=1}^k Q_{\beta_j}(t, q).$$

3.4. Decomposition and equivariant homology. A composition has at most one odd part if and only if all adjacent products in all of its coarsenings are nonzero. Indeed, if there are at least two odd parts, choose two consecutive odd parts; every intervening part is even, and absorbing those even parts into either side produces adjacent odd parts in a coarsening. Hence every iterated decoration product attached to a coarsening of such a composition is nonzero. In particular, the condition holds for every zigzag composition δ_n .

Fix $\alpha \models n$ with at most one odd part and let $\theta = (\sigma, \tau) \in \mathcal{D}_n$. For $c \in D(\alpha)$, call c a *factorization cut* of θ if

- (a) $\sigma([c^+]) = [c^+]$; and
- (b) every nonroot cycle of τ lies entirely in $[c^-]$ or entirely in $\{c^- + 1, \dots, n^-\}$, while the root cycle $(*, a_1, \dots, a_u)$ has all letters at most c^- before all letters greater than c^- .

The two sides of the cut cannot both have odd size, so these conditions are equivalent to the existence of unique decorations $\theta_L \in \mathcal{D}_c$ and $\theta_R \in \mathcal{D}_{n-c}$ such that $\theta = \theta_L \diamond \theta_R$. Set

$$F_\alpha(\theta) := \{c \in D(\alpha) : c \text{ is a factorization cut of } \theta\}, \quad \gamma_\alpha(\theta) := \gamma_{F_\alpha(\theta)}.$$

For the ordinary component, condition (a) is direct-sum factorization at c^+ , also called shifted concatenation. Connected permutations and their factorization under this product are discussed in [10, Sections 3.1–3.2, especially Equation (34) and Proposition 3.3]. The lemma below combines this ordinary factorization with rooted factorization at the same set of cuts.

Lemma 3.9 (Simultaneous factorization of permutation decorations). *Let $\beta \succeq \alpha$. Iterated multiplication defines an injection*

$$\mu_\beta : \mathcal{D}_{\beta_1} \times \dots \times \mathcal{D}_{\beta_k} \longrightarrow \mathcal{D}_n, \quad (\theta_1, \dots, \theta_k) \longmapsto \theta_1 \diamond \dots \diamond \theta_k,$$

whose image is precisely $\{\theta \in \mathcal{D}_n : D(\beta) \subseteq F_\alpha(\theta)\}$.

Proof. Put $c_0 = 0$ and $c_j = \beta_1 + \dots + \beta_j$ for $1 \leq j \leq k$. Because β has at most one odd part, c_{j-1} and β_j cannot both be odd. Applying the ceiling and floor identities to $c_j = c_{j-1} + \beta_j$ therefore gives

$$c_j^+ - c_{j-1}^+ = \beta_j^+, \quad c_j^- - c_{j-1}^- = \beta_j^-.$$

Let

$$I_j = \{c_{j-1}^+ + 1, \dots, c_j^+\}, \quad J_j = \{c_{j-1}^- + 1, \dots, c_j^-\}.$$

Suppose first that every $c_1, \dots, c_{k-1}$ is a factorization cut of $\theta = (\sigma, \tau)$. The nested prefix conditions $\sigma([c_j^+]) = [c_j^+]$ imply that each difference interval I_j is σ -invariant. Restricting σ to I_j and translating this interval to $[\beta_j^+]$ uniquely recovers a permutation $\sigma_j \in \mathfrak{S}_{\beta_j^+}$.

For the rooted component, every nonroot cycle of τ is confined by all the boundaries c_j^- to a unique interval J_j . The conditions on the root cycle at the nested prefix cuts are jointly equivalent to the following order condition: whenever $i < j$, every letter of J_i in the root cycle precedes every letter of J_j in that cycle. Taking from the root cycle the subword formed by letters in J_j , together with the nonroot cycles contained in J_j , and translating J_j to $[\beta_j^-]$ therefore gives a unique $\tau_j \in \mathfrak{R}_{\beta_j^-}$. If the subword is empty, its root cycle is the one-cycle $(*)$. By construction,

$$(\sigma, \tau) = (\sigma_1, \tau_1) \diamond \cdots \diamond (\sigma_k, \tau_k).$$

Conversely, an iterated product has invariant ordinary intervals, confined nonroot cycles, and subwords from successive intervals in the required order, so every c_j is a factorization cut. The restrictions above also show that the local factors are unique. This proves both injectivity and the asserted image characterization simultaneously for all selected cuts. $\square$

Example 3.10 (A small zigzag example). Let $\alpha = \delta_4 = (2, 2)$. Its only cut is 2, while the ordinary and rooted alphabets both have size 2. Consider the total decorations

$$\theta_1 = (\text{id}_2, (*)(1)(2)), \quad \theta_2 = (\text{id}_2, (*, 2, 1)), \quad \theta_3 = ((12), (*)(1)(2)).$$

For θ_1 , the ordinary prefix $[1]$ is invariant, the two nonroot cycles lie on opposite sides of the cut, and the condition on the root cycle is vacuous. Hence $F_{\delta_4}(\theta_1) = \{2\}$, and its fiber is $C_\bullet((2, 2))$ in bidegree $(4, 0)$. For θ_2 , the ordinary condition still holds, but the letters in the root cycle occur in the order 2, 1; thus $F_{\delta_4}(\theta_2) = \emptyset$, and its fiber is $C_\bullet((4))$ in bidegree $(2, 0)$. For θ_3 , the transposition does not preserve $[1]$, so the fiber is again $C_\bullet((4))$, now in bidegree $(3, 1)$. The defects defined in (57) are respectively 0, 1, and 0.

Theorem 3.11 (Factorization and homology of the decorated complex). *Let $\alpha \models n$ be nonempty, with at most one odd part. There is a canonical bidegree-preserving, $\mathfrak{S}_n$ -equivariant chain isomorphism*

$$C_\bullet^{t,q}(\alpha) \cong \bigoplus_{\theta \in \mathcal{D}_n} C_\bullet(\gamma_\alpha(\theta)),$$

where the summand indexed by θ is placed in bidegree $(\deg_t \theta, \deg_q \theta)$. Moreover,

$$(24) \quad \text{ch}_{t,q} H_k(C_\bullet^{t,q}(\alpha)) = \sum_{\substack{\theta \in \mathcal{D}_n \\ |F_\alpha(\theta)| = k-1}} t^{\deg_t \theta} q^{\deg_q \theta} r_{\gamma_\alpha(\theta)}.$$

Thus every bigraded homology Frobenius characteristic admits a nonnegative expansion in ribbon Schur functions and is therefore Schur-positive. The unique nonzero homology module contributed by θ occurs in degree $k = |F_\alpha(\theta)| + 1$ and has dimension

$$\#\{\pi \in \mathfrak{S}_n : \text{Des}(\pi) = F_\alpha(\theta)\}.$$

Proof. By Lemma 3.9, the permutation decorations satisfy the factorization condition in Definition 3.5. Apply Lemma 3.6. The dimension statement is (13) for each summand. $\square$

Continue to assume that α has at most one odd part. For $S \subseteq D(\alpha)$, set

$$(25) \quad G_{\alpha,S}(t, q) := \sum_{S \subseteq T \subseteq D(\alpha)} (-1)^{|T|-|S|} \prod_i Q_{(\gamma_T)_i}(t, q).$$

Corollary 3.12 (Decomposition by exact cut sets). *Suppose that $\alpha \models n$ is nonempty and has at most one odd part. The polynomial in (25) belongs to $\mathbb{N}[t, q]$, and*

$$(26) \quad G_{\alpha,S}(t, q) = \sum_{\substack{\theta \in \mathcal{D}_n \\ F_\alpha(\theta) = S}} t^{\deg_t \theta} q^{\deg_q \theta},$$

$$(27) \quad \text{ch}_{t,q} H_k(C_\bullet^{t,q}(\alpha)) = \sum_{\substack{S \subseteq D(\alpha) \\ |S| = k-1}} G_{\alpha,S}(t, q) r_{\gamma_S}.$$

In top degree, writing $r = \ell(\alpha)$,

$$(28) \quad \text{ch}_{t,q} H_r(C_{\bullet}^{t,q}(\alpha)) = \left(\prod_{i=1}^r Q_{\alpha_i}(t, q) \right) r_{\alpha}.$$

For $\alpha = \delta_n$ with $n \geq 1$, this specializes to

$$\text{ch}_{t,q} H_{\ell(\delta_n)}(C_{\bullet}^{t,q}(\delta_n)) = t^{\lceil n/2 \rceil} (1+t)^{\lfloor n/2 \rfloor} r_{\delta_n},$$

which lies in q -degree zero.

Proof. Unique factorization and degree additivity show that the weight enumerator of the decorations satisfying $T \subseteq F_{\alpha}(\theta)$ is $\prod_i Q_{(\gamma_T)_i}(t, q)$. Boolean Möbius inversion gives (26); grouping (24) by exact cut set gives (27). At $S = D(\alpha)$, unique factorization along all cuts gives the weight enumerator $\prod_i Q_{\alpha_i}$ and hence (28). For δ_n , use $Q_2 = t(t+1)$ and $Q_1 = t$. $\square$

3.5. The cut coproduct and total-decoration fibers. The adjacent-merge differential admits a deconcatenation coproduct. A cut at a part boundary of α produces the corresponding prefix and suffix compositions, so the natural object is the family over all compositions and all finite label sets.

For a finite set I with $|I| = n$, let $\mathfrak{S}_I$ denote its symmetric group, and let $C_{\bullet}^{t,q}(\alpha)[I]$ denote the complex of Definition 3.8, with I in place of $[n]$. For the empty composition, set $C_0^{t,q}(\emptyset)[\emptyset] = \mathbb{K}$ and set all other terms equal to zero. Relabeling the blocks along a bijection of finite sets makes

$$\mathcal{C}_{\bullet}[I] := \begin{cases} \bigoplus_{\alpha \models |I|} C_{\bullet}^{t,q}(\alpha)[I], & I \neq \emptyset, \\ C_{\bullet}^{t,q}(\emptyset)[\emptyset], & I = \emptyset, \end{cases}$$

a species in bigraded chain complexes. For species $\mathcal{F}$ and $\mathcal{G}$, their Cauchy product is

$$(\mathcal{F} \cdot \mathcal{G})[I] = \bigoplus_{I=S \sqcup T} \mathcal{F}[S] \otimes \mathcal{G}[T],$$

where the sum is over ordered decompositions; see [1].

Let $\alpha = (\alpha_1, \dots, \alpha_r)$ and put

$$c_p = \alpha_1 + \dots + \alpha_p, \quad \alpha_{\leq p} = (\alpha_1, \dots, \alpha_p), \quad \alpha_{> p} = (\alpha_{p+1}, \dots, \alpha_r),$$

for $0 \leq p \leq r$, with $c_0 = 0$ and with empty endpoint compositions. Let $I = S \sqcup T$ with $|S| = c_p$. On a basis element

$$w = (B_1, \dots, B_k; \xi_1, \dots, \xi_k) \in C_k^{t,q}(\alpha)[I],$$

define

$$(29) \quad \Delta_{S,T}^p(w) = \begin{cases} (B_1, \dots, B_j; \xi_1, \dots, \xi_j) \otimes (B_{j+1}, \dots, B_k; \xi_{j+1}, \dots, \xi_k), & \text{if } S = B_1 \sqcup \dots \sqcup B_j \text{ for some } 0 \leq j \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

The empty prefix and suffix are interpreted as the empty bar. In the first case, the two bars lie in $C_{\bullet}^{t,q}(\alpha_{\leq p})[S]$ and $C_{\bullet}^{t,q}(\alpha_{> p})[T]$, respectively.

For every ordered decomposition $I = S \sqcup T$, define the component $\Delta_{S,T} : \mathcal{C}_{\bullet}[I] \rightarrow \mathcal{C}_{\bullet}[S] \otimes \mathcal{C}_{\bullet}[T]$ on the summand indexed by α by

$$\Delta_{S,T}|_{C_{\bullet}^{t,q}(\alpha)[I]} = \begin{cases} \Delta_{S,T}^p, & |S| = c_p \text{ for some } 0 \leq p \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

The index p , when it exists, is unique. Let $\Delta_I = \bigoplus_{I=S \sqcup T} \Delta_{S,T}$. Let $\mathbf{1}$ be the unit species, equal to $\mathbb{K}$ on the empty set and zero otherwise, and let $\epsilon : \mathcal{C}_{\bullet} \rightarrow \mathbf{1}$ be the identity on $\mathcal{C}_{\bullet}[\emptyset] = \mathbb{K}$ and zero on nonempty sets.

Proposition 3.13 (Species-level cut coproduct). *Each map $\Delta_{S,T}^p$ is natural under relabeling, preserves the two weight degrees, and is a degree-zero chain map*

$$(30) \quad \Delta_{S,T}^p : C_k^{t,q}(\alpha)[I] \longrightarrow \bigoplus_{a+b=k} C_a^{t,q}(\alpha_{\leq p})[S] \otimes C_b^{t,q}(\alpha_{> p})[T],$$

where the target has the usual tensor-product differential. The natural transformation

$$\Delta : \mathcal{C}_\bullet \longrightarrow \mathcal{C}_\bullet \cdot \mathcal{C}_\bullet$$

is coassociative and counital. Explicitly, for every ordered decomposition $I = R \sqcup S \sqcup T$,

$$(31) \quad (\Delta_{R,S} \otimes \text{id}) \Delta_{R \sqcup S, T} = (\text{id} \otimes \Delta_{S,T}) \Delta_{R, S \sqcup T}.$$

The endpoint components satisfy

$$(32) \quad (\epsilon \otimes \text{id}) \Delta_{\emptyset, I} = \text{id}_{\mathcal{C}_\bullet[I]}, \quad (\text{id} \otimes \epsilon) \Delta_{I, \emptyset} = \text{id}_{\mathcal{C}_\bullet[I]}.$$

Thus $(\mathcal{C}_\bullet, \Delta, \epsilon)$ is a counital differential graded comonoid in the Cauchy monoidal category of species.

Proof. Suppose first that S is not a union of initial blocks of w . Merging adjacent blocks deletes a block boundary and cannot create such an initial union, so both sides of the chain-map identity vanish on w .

Now suppose that $S = B_1 \sqcup \cdots \sqcup B_j$. A face with index $i < j$ acts in the left tensor factor. A face with index $i > j$ acts in the right tensor factor, and its tensor-product sign is

$$(-1)^j (-1)^{i-j-1} = (-1)^{i-1},$$

the sign of the same face in the source. The face with index $i = j$ merges a block in S with a block in T . Its image under $\Delta_{S,T}^p$ is zero, and there is no corresponding term in the tensor-product differential. This proves (30); naturality and bidegree preservation are immediate.

For $I = R \sqcup S \sqcup T$, restrict both sides of (31) to a composition summand. They vanish unless R and $R \sqcup S$ have sizes equal to two part boundaries of that composition and are unions of the corresponding initial blocks. When these conditions hold, both sides return the same three consecutive bars. This proves coassociativity. The two endpoint cuts give (32). $\square$

For a nonempty composition $\alpha = (\alpha_1, \dots, \alpha_r)$, define its reduced cut coproduct by omitting the endpoint components:

$$(33) \quad \begin{aligned} \bar{\Delta}_{\alpha, I} : C_{\bullet}^{t,q}(\alpha)[I] &\longrightarrow \bigoplus_{p=1}^{r-1} \bigoplus_{\substack{I=S \sqcup T \\ |S|=c_p}} C_{\bullet}^{t,q}(\alpha_{\leq p})[S] \otimes C_{\bullet}^{t,q}(\alpha_{> p})[T], \\ \bar{\Delta}_{\alpha, I} &:= \bigoplus_{p=1}^{r-1} \bigoplus_{\substack{I=S \sqcup T \\ |S|=c_p}} \Delta_{S,T}^p. \end{aligned}$$

When $r = 1$, the target is zero and so is the map.

Assume from now on that α has at most one odd part. For $\theta \in \mathcal{D}_n$, let $C_{\bullet}^{t,q}(\alpha; \theta)[I]$ denote the subcomplex spanned by basis elements with total decoration θ .

Theorem 3.14 (Coproduct compatibility of total decorations). *Let $\alpha \models n$ have at most one odd part, set $r = \ell(\alpha)$, let $1 \leq p < r$, and put $c = c_p$. For every $I = S \sqcup T$ with $|S| = c$ and every $\theta \in \mathcal{D}_n$, the following hold.*

(i) *If $c \notin F_\alpha(\theta)$, then*

$$\Delta_{S,T}^p(C_{\bullet}^{t,q}(\alpha; \theta)[I]) = 0.$$

(ii) *If $c \in F_\alpha(\theta)$, write the unique factorization as $\theta = \theta_L \diamond \theta_R$, with $\theta_L \in \mathcal{D}_c$ and $\theta_R \in \mathcal{D}_{n-c}$. Then*

$$(34) \quad \Delta_{S,T}^p(C_{\bullet}^{t,q}(\alpha; \theta)[I]) \subseteq C_{\bullet}^{t,q}(\alpha_{\leq p}; \theta_L)[S] \otimes C_{\bullet}^{t,q}(\alpha_{> p}; \theta_R)[T].$$

Moreover,

$$(35) \quad F_\alpha(\theta) = F_{\alpha_{\leq p}}(\theta_L) \sqcup \{c\} \sqcup (c + F_{\alpha_{> p}}(\theta_R)),$$

$$(36) \quad \gamma_\alpha(\theta) = \gamma_{\alpha_{\leq p}}(\theta_L) \cdot \gamma_{\alpha_{> p}}(\theta_R).$$

Here $c + U = \{c + u : u \in U\}$ and $\cdot$ denotes concatenation of compositions.

For each $c_p \in F_\alpha(\theta)$, denote the corresponding factors by $\theta_L^{(p)}$ and $\theta_R^{(p)}$. Then

$$(37) \quad \overline{\Delta}_{\alpha,I}(C_{\bullet}^{t,q}(\alpha;\theta)[I]) \subseteq \bigoplus_{\substack{1 \leq p < \ell(\alpha) \\ c_p \in F_\alpha(\theta)}} \bigoplus_{\substack{I=S \sqcup T \\ |S|=c_p}} C_{\bullet}^{t,q}(\alpha_{\leq p}; \theta_L^{(p)})[S] \otimes C_{\bullet}^{t,q}(\alpha_{> p}; \theta_R^{(p)})[T].$$

Thus the reduced coproduct respects the total-decoration decomposition, with the decoration index split by unique factorization.

Proof. A basis element in the θ -fiber has block type β satisfying $D(\beta) \subseteq F_\alpha(\theta)$ by Lemma 3.9. If its image under $\Delta_{S,T}^p$ is nonzero, then $c \in D(\beta)$, which proves (i).

Suppose $c \in F_\alpha(\theta)$. If a bar splits at c , the products of its left and right local decorations factor θ at c . Uniqueness in Lemma 3.9 identifies them with θ_L and θ_R , proving (34).

A cut $d < c$ factors θ if and only if simultaneous factorization at d and c exists. By associativity and uniqueness, this is equivalent to factorization of θ_L at d . Similarly, a cut $d > c$ factors θ if and only if θ_R factors at $d - c$. This proves (35); taking cut sets gives (36). Summing the component inclusions over all internal cuts gives (37). $\square$

At the level of homology, the aggregate label-set component at a factorization cut is injective. Its cokernel is determined by the ribbon product rule.

For nonempty compositions $\lambda = (\lambda_1, \dots, \lambda_a)$ and $\mu = (\mu_1, \dots, \mu_b)$, write

$$\lambda \odot \mu = (\lambda_1, \dots, \lambda_{a-1}, \lambda_a + \mu_1, \mu_2, \dots, \mu_b)$$

for their near-concatenation.

Corollary 3.15 (Cut-induced ribbon injection). *Retain the hypotheses and notation of Theorem 3.14, assume $c \in F_\alpha(\theta)$, and set*

$$\lambda = \gamma_{\alpha_{\leq p}}(\theta_L), \quad \mu = \gamma_{\alpha_{> p}}(\theta_R).$$

Let

$$\Delta_c = \bigoplus_{\substack{S \subseteq I \\ |S|=c}} \Delta_{S,I \setminus S}^p.$$

Under the Künneth identification, the induced homology map is an injective $\mathfrak{S}_I$ -map

$$(38) \quad H(\Delta_c) : H_{\ell(\lambda) + \ell(\mu)}(C_{\bullet}^{t,q}(\alpha;\theta)[I]) \hookrightarrow \bigoplus_{\substack{S \subseteq I \\ |S|=c}} H_{\ell(\lambda)}(C_{\bullet}^{t,q}(\alpha_{\leq p}; \theta_L)[S]) \otimes H_{\ell(\mu)}(C_{\bullet}^{t,q}(\alpha_{> p}; \theta_R)[I \setminus S]).$$

All terms lie in bidegree $\text{bideg}(\theta) = \text{bideg}(\theta_L) + \text{bideg}(\theta_R)$. If $Q_{\theta,c}$ denotes the cokernel, then

$$(39) \quad \text{ch}_{t,q} Q_{\theta,c} = t^{\deg_t \theta} q^{\deg_q \theta} r_{\lambda \odot \mu}.$$

Thus $Q_{\theta,c}$ is noncanonically isomorphic to the classical ribbon representation indexed by $\lambda \odot \mu$, placed in the bidegree of θ .

Proof. The complexes are finite-dimensional over a field, so the Künneth theorem identifies the homology of each tensor product with the tensor product of the factor homologies. Both factors have homology concentrated in their top degrees by Proposition 3.4.

By (35), the source fiber is the classical ribbon complex for $\lambda \cdot \mu$. Its top chain group has block type $\lambda \cdot \mu$. Every top basis element has a unique prefix union of size c , and cutting there gives a pair of top basis elements of types λ and μ . Conversely, concatenating such a pair recovers the source basis element, and unique factorization recovers the local decorations. Therefore Δ_c is a bijection on top chain groups.

The target chain complex has no group above total degree $\ell(\lambda) + \ell(\mu)$. Hence a top cycle whose image is zero in homology already has zero image as a chain. The top-chain bijection proves injectivity. The target is the induced product of the two factor modules, so its Frobenius characteristic is $r_\lambda r_\mu$, while the source has character $r_{\lambda \cdot \mu}$. The ribbon product rule

$$r_\lambda r_\mu = r_{\lambda \cdot \mu} + r_{\lambda \odot \mu}$$

gives (39). In characteristic zero, the Frobenius characteristic determines the isomorphism class of a finite $\mathfrak{S}_I$ -module. $\square$

Corollary 3.16 (Cut-primitive homology). *Let $\alpha \models n$ have at most one odd part. Define*

$$(40) \quad \text{Prim}_{\text{cut}} H(C_{\bullet}^{t,q}(\alpha)) := \ker H(\overline{\Delta}_{\alpha,[n]}).$$

If $\ell(\alpha) = 1$, the reduced coproduct is zero by convention. Then

$$(41) \quad \text{Prim}_{\text{cut}} H(C_{\bullet}^{t,q}(\alpha)) = H_1(C_{\bullet}^{t,q}(\alpha)),$$

and

$$(42) \quad \text{ch}_{t,q} \text{Prim}_{\text{cut}} H(C_{\bullet}^{t,q}(\alpha)) = G_{\alpha,\emptyset}(t, q) r_{(n)}.$$

Equivalently, a total-decoration ribbon summand is cut-primitive exactly when its factorization-cut set is empty.

Proof. If $F_{\alpha}(\theta) = \emptyset$, part (i) of [Theorem 3.14](#) makes every internal component of the coproduct zero on the θ -fiber. If $F_{\alpha}(\theta) \neq \emptyset$, choose $c \in F_{\alpha}(\theta)$. The aggregate component at c is injective on the homology of that fiber by [Corollary 3.15](#). The target decomposes by the unique factor pair, so images from distinct total decorations cannot cancel. Thus the kernel of $H(\overline{\Delta}_{\alpha,[n]})$ is precisely the direct sum of the fibers with empty factorization-cut set. By [Theorem 3.11](#), these are exactly the fibers in homological degree one. Grouping them by the empty exact cut set and applying [Corollary 3.12](#) proves (41) and (42). $\square$

Remark 3.17 (Scope of the comonoid structure). The coproduct sends a composition to a prefix and a suffix, so an individual summand $C_{\bullet}^{t,q}(\alpha)$ is not a subcomonoid. The comonoid is the species $\mathcal{C}_{\bullet}$ obtained by summing over all compositions. The algebra A^{dec} organizes multiplication of local decorations, but no compatible product on the labelled complexes is defined here; accordingly, no Hopf-monoid structure is asserted.

3.6. Bigraded Euler characteristic. Since $Q_m(t, q)/m! = h_m(t, q)$, combining (23) with the ribbon coarsening formula gives, for every nonempty composition $\alpha \models n$,

$$(43) \quad \sum_{k=1}^{\ell(\alpha)} (-1)^{\ell(\alpha)-k} \Phi_{\alpha,k}(t, q) = n! Z_{\alpha}(t, q).$$

Thus $n! Z_{\alpha}(t, q)$ is the shifted bigraded Euler characteristic of the chain groups. When α has at most one odd part, Euler–Poincaré and [Theorem 3.11](#) also express it as the alternating sum of the explicitly determined homology groups.

Specializing $q = -1$ takes the signed difference between even and odd q -degrees. For zigzag compositions, the next section identifies this specialization with $n! \Omega(P_n; t)$ and retains the reflection-length variable.

4. ALTERNATING FENCES AND BIVARIATE POSITIVITY

We first identify the one-variable specialization with the alternating-fence order polynomial and then prove coefficientwise positivity for its bivariate refinement.

4.1. The alternating-fence specialization. For $n \geq 1$, let P_n be the alternating fence on $\{x_1, \dots, x_n\}$, with cover relations

$$x_1 < x_2 > x_3 < x_4 > \dots.$$

For every positive integer m , its order polynomial $\Omega(P_n; m)$ counts weakly order-preserving maps $f : P_n \rightarrow [m]$ satisfying $x \leq_P y \Rightarrow f(x) \leq f(y)$. Put $\Omega(P_0; t) = 1$.

Set

$$b_a(t) = \binom{t+a}{2a} \quad (a \geq 0), \quad d_a(t) = \binom{t+a}{2a+1} \quad (a \geq 0).$$

Thus $b_0(t) = 1$ and $d_0(t) = t$.

For later use, set

$$E_N(t, q) = Z_{\delta_{2N}}(t, q), \quad O_N(t, q) = Z_{\delta_{2N+1}}(t, q).$$

The coarsening formula gives

$$(44) \quad E_N(t, q) = \sum_{\alpha \models N} (-1)^{N-\ell(\alpha)} \prod_{j=1}^{\ell(\alpha)} h_{2\alpha_j}(t, q),$$

and hence

$$(45) \quad E_N(t, q) = \sum_{a=1}^N (-1)^{a+1} h_{2a}(t, q) E_{N-a}(t, q), \quad E_0(t, q) = 1.$$

Similarly, grouping a coarsening of $(2^N, 1)$ by the number a of twos merged with its final one gives

$$(46) \quad O_N(t, q) = \sum_{a=0}^N (-1)^a h_{2a+1}(t, q) E_{N-a}(t, q).$$

The following known identification is an instance of the Kreweras determinant.

Proposition 4.1 (Alternating-fence specialization). *For every $n \geq 0$,*

$$(47) \quad \Omega(P_n; t) = Z_{\delta_n}(t, -1) = \phi_t(R_{\delta_n}).$$

Proof. The case $n = 0$ is immediate. Kreweras's determinant, in the form [6, Proposition 2.2], applies to the two zigzag ribbons in [6, Equation (5.1)]. Their cell posets are dual to P_{2N} and P_{2N+1} , respectively, and duality preserves the weak order polynomial. Substitution in the determinant gives, with $\Delta_0(t) = 1$ and $b_a = 0$ for $a < 0$,

$$(48) \quad \Omega(P_{2N}; t) = \Delta_N(t), \quad \Delta_N(t) = \det[b_{j-i+1}(t)]_{i,j=1}^N.$$

For the odd ribbon, the same substitution gives the Hessenberg determinant

$$\Omega(P_{2N+1}; t) = \det \begin{pmatrix} b_1 & b_2 & \cdots & b_N & d_N \\ 1 & b_1 & \cdots & b_{N-1} & d_{N-1} \\ 0 & 1 & \cdots & b_{N-2} & d_{N-2} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & d_0 \end{pmatrix}.$$

Expansion along the last column gives

$$(49) \quad \Omega(P_{2N+1}; t) = \sum_{a=0}^N (-1)^a d_a(t) \Delta_{N-a}(t).$$

Expanding (48) along the first row yields

$$\Delta_N(t) = \sum_{a=1}^N (-1)^{a+1} b_a(t) \Delta_{N-a}(t).$$

By (7), this is exactly (45) at $q = -1$, so $\Delta_N(t) = E_N(t, -1)$. Substituting this equality and (8) into (49) gives (46) at $q = -1$. Hence $\Omega(P_n; t) = Z_{\delta_n}(t, -1)$ in both parities. $\square$

4.2. Generating functions. Introduce

$$\mathcal{B}_{t,q}(x) = \sum_{a \geq 0} h_{2a}(t, q) x^a, \quad \mathcal{O}_{t,q}(x) = \sum_{a \geq 0} h_{2a+1}(t, q) x^a.$$

We use the rising Pochhammer symbol

$$(a)_n = a(a+1) \cdots (a+n-1), \quad (a)_0 = 1,$$

and

$${}_2F_1(A, B; C; x) = \sum_{n \geq 0} \frac{(A)_n (B)_n}{(C)_n n!} x^n.$$

Lemma 4.2 (Generating functions for the zigzag compositions). *In $\mathcal{R}[[x]]$,*

$$\begin{aligned} \sum_{N \geq 0} E_N(t, q) x^N &= \frac{1}{\mathcal{B}_{t,q}(-x)}, \\ \sum_{N \geq 0} O_N(t, q) x^N &= \frac{\mathcal{O}_{t,q}(-x)}{\mathcal{B}_{t,q}(-x)}. \end{aligned}$$

Moreover, in $\mathbb{Q}[t, q, q^{-1}][[x]]$,

$$(50) \quad \mathcal{B}_{t,q}(-x) = {}_2F_1\left(\frac{t}{q}, t+1; \frac{1}{2}; -\frac{qx}{4}\right).$$

Every coefficient of these identities belongs to $\mathbb{Q}[t, q]$. Hence they have a unique coefficientwise specialization at $q = 0$, even though the displayed hypergeometric parameters contain q^{-1} .

Proof. Multiplying (45) by x^N and summing gives $\sum E_N x^N = \mathcal{B}_{t,q}(-x)^{-1}$. Equation (46) then gives $\sum O_N x^N = \mathcal{O}_{t,q}(-x) \sum E_N x^N$. The Pochhammer and duplication identities give, coefficientwise,

$$h_{2a}(t, q) = \frac{(t/q)_a (t+1)_a}{(1/2)_a a!} \left(\frac{q}{4}\right)^a.$$

Summation proves (50). Finally, (4) places both kernel series in $\mathbb{Q}[t, q][[x]]$; the even kernel has constant term 1. Its inverse lies there as well, which proves the assertion at $q = 0$. $\square$

4.3. Coefficientwise positivity. Ribbon inclusion–exclusion introduces alternating signs. After the substitution $q = -u$, multiplying the zigzag evaluation by $n!$ gives a polynomial with nonnegative coefficients.

Theorem 4.3 (Bivariate coefficientwise positivity). *For every $n \geq 0$,*

$$(51) \quad n! Z_{\delta_n}(t, -u) \in \mathbb{N}[t, u].$$

Equivalently, whenever $[t^i q^j] n! Z_{\delta_n}(t, q) \neq 0$, its sign is $(-1)^j$.

At $u = 1$, Theorem 4.3 specializes to coefficientwise nonnegativity of $n! \Omega(P_n; t)$, also implied by Kahane’s permutation-statistic theorem for arbitrary fence order polynomials [12]. The theorem above also records the q -degree $m^+ - c(\sigma)$, which is lost when $u = 1$. Its proof uses the following recurrence for the even generating function.

Lemma 4.4 (Positive Riccati recurrences). *Put*

$$B(x) = \mathcal{B}_{t,-u}(-x), \quad A(x) = -\frac{B'(x)}{B(x)} = \sum_{n \geq 0} a_n x^n, \quad e_n = 4a_{n+1} - ua_n.$$

Then

$$a_n, e_n \in \mathbb{Q}_{\geq 0}[t, u] \quad (n \geq 0).$$

Proof. By Lemma 4.2,

$$B(x) = {}_2F_1\left(-\frac{t}{u}, t+1; \frac{1}{2}; \frac{ux}{4}\right).$$

Gauss’s differential equation [15, Eq. (15.10.1)], after the substitution $z = ux/4$, becomes

$$(52) \quad (4x - ux^2)B'' + (2 - (tu - t + 2u)x)B' + t(t+1)B = 0.$$

Although the displayed hypergeometric parameters contain u^{-1} , (52) is an identity in $\mathbb{Q}[t, u][[x]]$: it also follows directly from the ratio of consecutive coefficients of B . In particular, it remains valid at $u = 0$.

Substitution of $B' = -AB$ into (52) gives

$$(53) \quad (4x - ux^2)A' = (4x - ux^2)A^2 - (2 - (tu - t + 2u)x)A + t(t+1).$$

Set

$$\kappa_m = \sum_{i=0}^m a_i a_{m-i} \quad (m \geq 0), \quad \kappa_m = 0 \quad (m < 0).$$

The constant term in (53) is

$$a_0 = \frac{t(t+1)}{2}.$$

For $n \geq 1$, its coefficient of x^n gives

$$(54) \quad (4n+2)a_n = 4\kappa_{n-1} - u\kappa_{n-2} + (tu - t + (n+1)u)a_{n-1}.$$

Using the definition of e_i , we may rewrite these terms as

$$4\kappa_{n-1} - u\kappa_{n-2} = \sum_{i=0}^{n-2} a_i e_{n-2-i} + 4a_0 a_{n-1},$$

so (54) becomes

$$(55) \quad (4n+2)a_n = \sum_{i=0}^{n-2} a_i e_{n-2-i} + (2t^2 + t + tu + (n+1)u)a_{n-1}.$$

Here and below an empty sum is zero. Applying (54) with index $n+1$, multiplying by four, and subtracting $(4n+6)ua_n$ gives

$$(4n+6)e_n = 4(4\kappa_n - u\kappa_{n-1}) + (4tu - 4t + 2u)a_n.$$

Since

$$4\kappa_n - u\kappa_{n-1} = \sum_{i=0}^{n-1} a_i e_{n-1-i} + 4a_0 a_n,$$

and $4a_0 = 2t(t+1)$, we obtain

$$(56) \quad (4n+6)e_n = 4 \sum_{i=0}^{n-1} a_i e_{n-1-i} + (8t^2 + 4t + 4tu + 2u)a_n \quad (n \geq 0).$$

Now $a_0 \in \mathbb{Q}_{\geq 0}[t, u]$; (55) at $n=1$ gives $a_1 \geq 0$, and (56) at $n=0$ gives $e_0 \geq 0$. If $a_0, \dots, a_n$ and $e_0, \dots, e_{n-1}$ are coefficientwise nonnegative for $n \geq 1$, then (55) at $n+1$ gives $a_{n+1} \geq 0$, while (56) at n gives $e_n \geq 0$. Simultaneous induction proves the lemma. $\square$

Proof of Theorem 4.3. Let

$$F_{\text{even}}(x) = \sum_{N \geq 0} Z_{\delta_{2N}}(t, -u)x^N, \quad F_{\text{odd}}(x) = \sum_{N \geq 0} Z_{\delta_{2N+1}}(t, -u)x^N.$$

By Lemma 4.2, $F_{\text{even}} = 1/B$. Hence $F'_{\text{even}} = AF_{\text{even}}$ and $F_{\text{even}}(0) = 1$. If $F_{\text{even}} = \sum_{N \geq 0} f_N x^N$, then

$$(N+1)f_{N+1} = \sum_{i+j=N} a_i f_j.$$

The lemma shows inductively that every f_N lies in $\mathbb{Q}_{\geq 0}[t, u]$.

For the odd series, set $J(x) = \mathcal{O}_{t,-u}(-x)$. The local identity

$$(2m+1)h_{2m+1}(t, -u) = (t - mu)h_{2m}(t, -u)$$

gives, after summation with the kernel signs,

$$2xJ'(x) + J(x) = tB(x) - uxB'(x).$$

Since $F_{\text{odd}} = J/B$ and $B'/B = -A$, this is equivalent to

$$2xF'_{\text{odd}}(x) + F_{\text{odd}}(x) = 2xA(x)F_{\text{odd}}(x) + t + uxA(x).$$

Writing $F_{\text{odd}} = \sum_{N \geq 0} o_N x^N$, we obtain

$$o_0 = t, \quad (2N+1)o_N = 2 \sum_{i+j=N-1} a_i o_j + ua_{N-1} \quad (N \geq 1).$$

Another induction gives $o_N \in \mathbb{Q}_{\geq 0}[t, u]$. We have therefore proved

$$Z_{\delta_n}(t, -u) \in \mathbb{Q}_{\geq 0}[t, u] \quad (n \geq 0).$$

The case $n=0$ is immediate. For $n \geq 1$, put $Q_m(t, q) = m!h_m(t, q) \in \mathbb{Z}[t, q]$. The coarsening formula gives, for every nonempty composition $\alpha \models n$,

$$n!Z_\alpha(t, q) = \sum_{\beta \succeq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} \binom{n}{\beta_1, \dots, \beta_{\ell(\beta)}} \prod_i Q_{\beta_i}(t, q) \in \mathbb{Z}[t, q].$$

Applying this to $\alpha = \delta_n$ and combining it with the coefficientwise nonnegativity over $\mathbb{Q}$ proved above gives (51). $\square$

4.4. Factorization defect and the support region. We now connect the homology decomposition with the sign pattern in [Theorem 4.3](#). Put

$$\ell_n = \ell(\delta_n) = \left\lceil \frac{n}{2} \right\rceil, \quad H_k^{i,j}(n) = H_k(C_{\bullet}^{t,q}(\delta_n))_{(i,j)}.$$

For a total decoration $\theta = (\sigma, \tau) \in \mathcal{D}_n$, define its *factorization defect* by

$$(57) \quad \text{fd}_n(\theta) := c(\sigma) - |F_{\delta_n}(\theta)| - 1.$$

For a basis element of $C_{\bullet}^{t,q}(\delta_n)$, multiply its local decorations in block order and call the result its total decoration. This product is nonzero because every coarsening of δ_n has at most one odd part. For an integer d , let $C_{\bullet}^{t,q}(\delta_n; d)$ be the span of the basis elements whose total decoration has factorization defect d . We will show that these spaces are subcomplexes and vanish for $d < 0$. For $d \geq 0$, set

$$(58) \quad \mathcal{H}_{n,d}(t, u) := \sum_{\substack{\theta \in \mathcal{D}_n \\ \text{fd}_n(\theta) = d}} t^{\deg_t \theta} u^{\deg_q \theta} r_{\gamma_{\delta_n}(\theta)}.$$

This is a homogeneous symmetric function of degree n , presented as a nonnegative linear combination of ribbon Schur functions with coefficients in $\mathbb{N}[t, u]$.

Theorem 4.5 (Defect decomposition and support). *Let $n \geq 1$. Every total decoration satisfies $0 \leq \text{fd}_n(\theta) \leq \ell_n - 1$. The differential preserves total decoration and gives a canonical decomposition into bigraded $\mathfrak{S}_n$ -subcomplexes*

$$(59) \quad C_{\bullet}^{t,q}(\delta_n) = \bigoplus_{d=0}^{\ell_n-1} C_{\bullet}^{t,q}(\delta_n; d).$$

If $\theta = (\sigma, \tau)$ contributes to $H_k^{i,j}(n)$, then

$$(60) \quad \text{fd}_n(\theta) = \ell_n - k - j.$$

Consequently,

$$(61) \quad H_k^{i,j}(n) = 0 \quad \text{unless} \quad k + j \leq \ell_n \leq i + j.$$

For every $d \geq 0$,

$$(62) \quad \begin{aligned} \mathcal{H}_{n,d}(t, u) &= \sum_k \text{ch}_{t,u} H_k(C_{\bullet}^{t,q}(\delta_n; d)) \\ &= \sum_{\substack{k \geq 1, j \geq 0 \\ k+j=\ell_n-d}} u^j [q^j] \text{ch}_{t,q} H_k(C_{\bullet}^{t,q}(\delta_n)), \end{aligned}$$

$$(63) \quad \sum_k (-1)^{\ell_n-k} \text{ch}_{t,-u} H_k(C_{\bullet}^{t,q}(\delta_n; d)) = (-1)^d \mathcal{H}_{n,d}(t, u).$$

Here $\text{ch}_{t,u}$ and $\text{ch}_{t,-u}$ mean substitution of $q = u$ and $q = -u$, respectively. Summing over d gives

$$(64) \quad \sum_{k=1}^{\ell_n} (-1)^{\ell_n-k} \text{ch}_{t,-u} H_k(C_{\bullet}^{t,q}(\delta_n)) = \sum_{d=0}^{\ell_n-1} (-1)^d \mathcal{H}_{n,d}(t, u).$$

If $\dim_n(f) := n![p_1^n]f$ denotes the dimension of the virtual $\mathfrak{S}_n$ -module with Frobenius characteristic f , then

$$(65) \quad n!Z_{\delta_n}(t, -u) = \dim_n \left(\sum_{d=0}^{\ell_n-1} (-1)^d \mathcal{H}_{n,d}(t, u) \right).$$

Thus the bivariate positivity theorem states that the scalar dimension of an alternating sum of classes with nonnegative ribbon Schur expansions is coefficientwise nonnegative.

Proof. For a total decoration $\theta = (\sigma, \tau)$, put $k_\theta = |F_{\delta_n}(\theta)| + 1$. Factorization at these cuts partitions the ordinary alphabet $[\ell_n]$ into k_θ nonempty intervals, each invariant under σ . Every interval contains at least one cycle, so $c(\sigma) \geq k_\theta$ and $\text{fd}_n(\theta) = c(\sigma) - k_\theta \geq 0$. Also $c(\sigma) \leq \ell_n$ and $k_\theta \geq 1$, which gives the upper bound $\text{fd}_n(\theta) \leq \ell_n - 1$.

Every nonzero face multiplies two adjacent local decorations, so it preserves their iterated product and hence the total decoration. The external $\mathfrak{S}_n$ -action changes only block labels and preserves it as well. The spaces $C_{\bullet}^{t,q}(\delta_n; d)$ are therefore bigraded $\mathfrak{S}_n$ -subcomplexes, and the preceding bounds give (59). By Theorem 3.11, the fiber indexed by θ contributes in homological degree $k = k_\theta$. Since $j = \ell_n - c(\sigma)$, this proves (60) and the first inequality in (61). The other inequality follows from

$$i + j = c(\sigma) + c(\tau) - 1 + \ell_n - c(\sigma) = \ell_n + c(\tau) - 1 \geq \ell_n.$$

Formula (24), grouped by the value of fd_n , gives (62) and also shows that no negative-defect summand occurs. A summand in homological degree k and q -degree j has Euler sign

$$(-1)^{\ell_n - k} (-1)^j = (-1)^{\ell_n - k - j} = (-1)^{\text{fd}_n(\theta)}.$$

The sign is therefore constant on the homology of each defect subcomplex, which proves (63) and (64). Applying $\dim_n$ and using Euler–Poincaré together with (43) gives (65). $\square$

Remark 4.6 (A non-Schur-positive virtual class). The virtual Frobenius characteristic in (64) need not be coefficientwise Schur-positive. For $n = 4$, direct substitution in the exact-cut formula gives

$$\sum_{k=1}^2 (-1)^{2-k} \text{ch}_{t,-u} H_k(C_{\bullet}^{t,q}(\delta_4)) = t^2(t+1)^2 r_{(2,2)} + t(t+1)(u(t+2)-t)r_{(4)}.$$

Since $r_{(4)} = s_{(4)}$ and $r_{(2,2)} = s_{(3,1)} + s_{(2,2)}$, the coefficient of $s_{(4)}$ is $t(t+1)(u(t+2)-t)$ and is not coefficientwise nonnegative. Thus the virtual class itself need not be coefficientwise Schur-positive, although its dimension is coefficientwise nonnegative by Theorem 4.3.

The defect-zero layer is the edge $k + j = \ell_n$ of (61). It has an explicit Frobenius formula. If $\lambda = (\lambda_1, \dots, \lambda_r)$ is a composition, write

$$2\lambda = (2\lambda_1, \dots, 2\lambda_r), \quad 2\lambda - \mathbf{e}_r = (2\lambda_1, \dots, 2\lambda_{r-1}, 2\lambda_r - 1).$$

Corollary 4.7 (The defect-zero edge). *For $N \geq 1$,*

$$(66) \quad \mathcal{H}_{2N,0}(t, u) = \sum_{\lambda \models N} u^{N-\ell(\lambda)} \left(\prod_{a=1}^{\ell(\lambda)} (\lambda_a - 1)! (t)_{\lambda_a+1} \right) r_{2\lambda}.$$

For $N \geq 0$,

$$(67) \quad \mathcal{H}_{2N+1,0}(t, u) = \sum_{\lambda \models N+1} u^{N+1-\ell(\lambda)} \left(\prod_{a=1}^{\ell(\lambda)} (\lambda_a - 1)! \right) \left(\prod_{a=1}^{\ell(\lambda)-1} (t)_{\lambda_a+1} \right) (t)_{\lambda_{\ell(\lambda)}} r_{2\lambda - \mathbf{e}_{\ell(\lambda)}}.$$

Because the u -degree determines the homological degree on this edge, these two identities determine every representation with $k + j = \ell_n$.

Proof. Fix an exact cut set of size $k - 1$. In the even case it determines a composition $\lambda \models N$ of length k and the ribbon composition 2λ . The corresponding ordinary factor intervals have sizes $\lambda_1, \dots, \lambda_k$. Defect zero means that the ordinary permutation has exactly one cycle on each interval, giving $(\lambda_a - 1)!$ choices on the a -th interval. A single cycle has no proper invariant prefix, so these choices introduce no additional factorization cuts.

The rooted factor on the a -th interval is arbitrary and has cycle enumerator $(t)_{\lambda_a+1}$. Since the ordinary component has k cycles and gluing the k rooted factors decreases their total number of cycles by $k - 1$, the global t -weight is the product of these rooted cycle enumerators. The q -degree is $N - k$. Summing over all exact cut sets proves (66).

In the odd case, an exact cut set determines a composition $\lambda \models N + 1$ of length k and the ribbon composition $2\lambda - \mathbf{e}_k$. The ordinary interval sizes are again the λ_a . The first $k - 1$ rooted intervals have sizes λ_a , while the last has size $\lambda_k - 1$. Their cycle enumerators are respectively $(t)_{\lambda_a+1}$ and $(t)_{\lambda_k}$, and the q -degree is $N + 1 - k$. The same argument proves (67). $\square$

The endpoint $k = 1$ gives the maximal u -degree. Applying the dimension map to the single ribbon $r_{(n)} = h_n$ yields

$$(68) \quad [u^{N-1}](2N)! Z_{\delta_{2N}}(t, -u) = (N-1)!(t)_{N+1} \quad (N \geq 1),$$

$$(69) \quad [u^N](2N+1)! Z_{\delta_{2N+1}}(t, -u) = N!(t)_{N+1} \quad (N \geq 0).$$

4.5. Logarithmic positivity of the even series. Write

$$F_{\text{even}}(x) := \sum_{N \geq 0} Z_{\delta_{2N}}(t, -u)x^N = B(x)^{-1}.$$

The logarithmic derivative used in the positivity proof also controls the logarithm of this series.

Proposition 4.8 (Positive logarithmic coefficients). *For $N \geq 1$, define*

$$(70) \quad \Lambda_N(t, u) = \frac{(2N)!}{N} a_{N-1}(t, u),$$

where $-B'(x)/B(x) = \sum_{r \geq 0} a_r(t, u)x^r$. Then

$$(71) \quad \Lambda_N(t, u) \in \mathbb{N}[t, u], \quad F_{\text{even}}(x) = \exp \left(\sum_{N \geq 1} \Lambda_N(t, u) \frac{x^N}{(2N)!} \right).$$

At $u = 1$,

$$(72) \quad \Lambda_N(t, 1) = \frac{(2N)!}{N} \Omega(C_{2N}^\circ; t),$$

where C_{2N}° is the alternating circular fence with sign word $(+, -)^N$, with C_2° understood as the two-element chain.

Proof. Since $F_{\text{even}} = B^{-1}$, integration gives

$$\log F_{\text{even}}(x) = \sum_{N \geq 1} \frac{a_{N-1}(t, u)}{N} x^N,$$

which proves the exponential formula. By Lemma 4.4, the polynomials Λ_N have nonnegative rational coefficients.

For integrality, put

$$M_{2r}(t, u) = (2r)! Z_{\delta_{2r}}(t, -u), \quad M_{2r+1}(t, u) = 0, \quad M_0(t, u) = 1.$$

Theorem 4.3 gives $M_s \in \mathbb{Z}[t, u]$. The moment-cumulant formula applied to

$$F_{\text{even}}(z^2) = \sum_{s \geq 0} M_s(t, u) \frac{z^s}{s!}$$

gives

$$\Lambda_N(t, u) = \sum_{\substack{\mathcal{Q} \text{ a set partition of } [2N] \\ |D| \text{ even for every } D \in \mathcal{Q}}} (-1)^{|\mathcal{Q}|-1} (|\mathcal{Q}|-1)! \prod_{D \in \mathcal{Q}} M_{|D|}(t, u).$$

Thus $\Lambda_N \in \mathbb{Z}[t, u]$, and coefficientwise nonnegativity over $\mathbb{Q}$ proves $\Lambda_N \in \mathbb{N}[t, u]$.

It remains to prove (72). For a positive integer m , let $K_m = [\min(i, j)]_{i, j \in [m]}$. A principal minor indexed by $1 \leq s_1 < \dots < s_r \leq m$ equals

$$s_1(s_2 - s_1) \cdots (s_r - s_{r-1}).$$

With $d_1 = s_1$, $d_i = s_i - s_{i-1}$ for $2 \leq i \leq r$, and $d_{r+1} = m + 1 - s_r$, the sum of these minors is

$$[z^{m+1}] \left(\frac{z}{(1-z)^2} \right)^r \frac{z}{1-z} = \binom{m+r}{2r}.$$

It follows that

$$B(x)|_{t=m, u=1} = \det(I_m - xK_m),$$

and hence

$$-\frac{B'(x)}{B(x)} \Big|_{t=m, u=1} = \sum_{N \geq 1} \text{tr}(K_m^N) x^{N-1}.$$

To evaluate the trace, assign values $y_1, \dots, y_N \in [m]$ to the maximal elements of C_{2N}° . The minimal element between y_{i-1} and y_i has $\min(y_{i-1}, y_i)$ possible values, with cyclic indices. Therefore

$$\Omega(C_{2N}^\circ; m) = \text{tr}(K_m^N).$$

Both sides are polynomials in m , so interpolation gives $a_{N-1}(t, 1) = \Omega(C_{2N}^\circ; t)$ and proves (72). $\square$

5. CONCLUSION

Total permutation decorations provide a common structure for the homological and enumerative results. Their simultaneous factorization cuts identify the classical ribbon complex in each fiber and are respected by the comonoid cut coproduct on labelled complexes. This yields all bigraded homology representations, cut-induced injections at factorization cuts, and the identification of cut primitives with H_1 .

For zigzag compositions, the excess of ordinary cycles over factorization intervals is the nonnegative defect $\lceil n/2 \rceil - k - j$. It determines the homological support, fixes the Euler sign on each defect layer, and leads to explicit Frobenius formulas on the defect-zero edge. The Riccati recurrences prove coefficientwise positivity of the reflection-length refinement and its logarithmic coefficients. The $n = 4$ virtual class shows that the scalar positivity does not generally lift to coefficientwise Schur positivity before taking dimensions.

ACKNOWLEDGMENTS

The author thanks Professor Yanpeng Li for helpful discussions and comments on the organization of the proofs.

USE OF GENERATIVE AI

Generative-AI tools (OpenAI ChatGPT and Codex, August 2026) were used for language editing, notation and cross-reference checks, bibliographic verification, and drafting finite verification scripts. No computer output is used as a proof. The author independently checked every mathematical statement and assumes full responsibility for the manuscript.

REFERENCES

- [1] M. Aguiar and S. Mahajan. *Monoidal Functors, Species and Hopf Algebras*, volume 29 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2010. ISBN 978-0-8218-4776-3. doi: 10.1090/crmm/029.
- [2] M. Aguiar, N. Bergeron, and F. Sottile. Combinatorial Hopf algebras and generalized Dehn–Sommerville relations. *Compositio Mathematica*, 142(1):1–30, 2006. doi: 10.1112/S0010437X0500165X.
- [3] A. Almousa and B. Lu. Ribbon complexes for the 0-Hecke algebra, 2026. arXiv:2601.13324v1.
- [4] F. Bergeron and D. Krob. Acyclic complexes related to noncommutative symmetric functions. *Journal of Algebraic Combinatorics*, 6(2):103–117, 1997. doi: 10.1023/A:1008622519966.
- [5] R. Ehrenborg and J. Jung. The topology of restricted partition posets. *Journal of Algebraic Combinatorics*, 37(4):643–666, 2013. doi: 10.1007/s10801-012-0379-8.
- [6] L. Ferroni, A. H. Morales, and G. Panova. Skew shapes, Ehrhart positivity and beyond, 2025. To appear in Proceedings of the London Mathematical Society; arXiv:2503.16403v3.
- [7] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh, and J.-Y. Thibon. Non-commutative symmetric functions. *Advances in Mathematics*, 112(2):218–348, 1995. doi: 10.1006/aima.1995.1032.
- [8] I. M. Gessel and C. Reutenauer. Counting permutations with given cycle structure and descent set. *Journal of Combinatorial Theory, Series A*, 64(2):189–215, 1993. doi: 10.1016/0097-3165(93)90095-P.
- [9] P. Hersh and S. Sundaram. Stability and ribbon bases for the rank-selected homology of geometric lattices, 2026. arXiv:2604.06479v2.
- [10] F. Hivert, J.-C. Novelli, and J.-Y. Thibon. Commutative combinatorial Hopf algebras. *Journal of Algebraic Combinatorics*, 28(1):65–95, 2008. doi: 10.1007/s10801-007-0077-0.
- [11] P. C. Huang. Bernstein transfers and greedy records for fence and circular-fence order polynomials, 2026. arXiv:2607.22767v2.
- [12] Y. Kahane. Combinatorial interpretation of the coefficients of the order polynomial of fence posets, 2026. arXiv:2607.11225v1.
- [13] G. Kreweras. Sur une classe de problèmes de dénombrement liés au treillis des partitions des entiers. *Cahiers du Bureau universitaire de recherche opérationnelle, Série Recherche*, 6:9–107, 1965. URL https://www.numdam.org/item/BURO_1965__6__9_0/.

- [14] J.-C. Novelli, J.-Y. Thibon, and F. Toumazet. A noncommutative cycle index and new bases of quasi-symmetric functions and noncommutative symmetric functions. *Annals of Combinatorics*, 24(3):557–576, 2020. doi: 10.1007/s00026-020-00504-5.
- [15] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST Handbook of Mathematical Functions*. Cambridge University Press, Cambridge, 2010. URL <https://dlmf.nist.gov/15>.
- [16] R. P. Stanley. Some aspects of groups acting on finite posets. *Journal of Combinatorial Theory, Series A*, 32(2):132–161, 1982. doi: 10.1016/0097-3165(82)90017-6.
- [17] R. P. Stanley. *Enumerative Combinatorics, Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, second edition, 2012. doi: 10.1017/CBO9781139058520.
- [18] K. VandeBogert. Ribbon Schur functors. *Algebra & Number Theory*, 19(4):771–834, 2025. doi: 10.2140/ant.2025.19.771.
- [19] M. L. Wachs. Poset topology: tools and applications. In *Geometric Combinatorics*, volume 13 of *IAS/Park City Mathematics Series*, pages 497–615. American Mathematical Society, Providence, RI, 2007. doi: 10.1090/pcms/013/09.

SCHOOL OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU 610064, CHINA
Email address: pyuyi233@gmail.com